On a problem from the Kourovka Notebook*

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Abstract

In this manuscript, a solution to Problem 18.91(b) in the Kourovka Notebook is given by proving the following theorem. Let P be a Sylow p-subgroup of a group G with $|P| = p^n$. Suppose that there is an integer k such that $1 < k < n$ and every subgroup of P of order p^k is S-propermutable in G, and also, in the case that $p = 2$, $k = 1$ and P is non-abelian, every cyclic subgroup of P of order 4 is S-propermutable in G. Then G is p-nilpotent.

Recall that a subgroup H of a group G is said to be *propermutable* (resp. *S-propermutable*) [2] in G, if there is a subgroup B of G such that $G = N_G(H)B$ and H permutes with every subgroup (resp. Sylow subgroup) of B. The aim of this manuscript is to give a solution to the following problem proposed by A. N. Skiba in the Kourovka Notebook.

Problem 1. (*see Problem* 18.91(*b*) *in* [1].) *Let P be a non-abelian Sylow* 2*-subgroup of a group G with $|P| = 2^n$. Suppose that there is an integer k such that $1 < k < n$ and every subgroup of P of order 2^k is propermutable in G, and also, in the case of $k = 1$, every cyclic subgroup of P of order* 4 *is propermutable in G. Is it true that then G is* 2*-nilpotent?*

A subgroup H of a group G is said to satisfy Π-*property* [3] in G if for every chief factor L/K of G, $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a $\pi(HK/K \cap L/K)$-number. Now we can establish the relationship between S-propermutable subgroups and subgroups which satisfy Π-property.

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Lemma 2. *If a p-subgroup H is S-propermutable in a group G, then H satisfies Π-property in G.*

Proof. In view of [2, Lemma 2.3(1)] and [3, Proposition 2.1(1)], we only need to prove that $|G : N_G(H \cap N)|$ is a p-number for any minimal normal subgroup N of G by induction. If N is abelian, then $|G : N_G(H \cap N)|$ is a p-number by [2, Lemma 2.3(4)]. We may, therefore, assume that N is non-abelian.

Now we shall prove that $H \cap N = 1$. As H is S-propermutable in G, G has a subgroup B such that $G = N_G(H)B$ and H permutes with every Sylow subgroup of B. Clearly, $H^G \leq HB$. If $H^G \cap N = 1$, then $H \cap N = 1$. Hence we may assume that $N \leq H^G \leq HB$. Then for any Sylow q-subgroup N_q of N with $q \neq p$, B has a Sylow q-subgroup B_q such that $N_q = (B_q)^h \cap N$ for some $h \in H$. It follows that $H(B_q)^h \cap N = (H \cap N)((B_q)^h \cap N) = (H \cap N)N_q$, and so $H \cap N$ permutes with N_q. Since $N \neq (H \cap N)N_q$, N has a proper normal subgroup L such that either $H \cap N \leq L$ or $N_q \leq L$. Then evidently, we have that $H \cap N \leq L$. Note that for any Sylow q-subgroup L_q of L, $H \cap N$ permutes with L_q. Repeating this argument, we can obtain that $H \cap N = 1$. The lemma is thus proved. $\square$

We arrive at the following main result.

Theorem 3. *Let P be a Sylow p-subgroup of a group G with $|P| = p^n$. Suppose that there is an integer k such that $1 < k < n$ and every subgroup of P of order p^k is S-propermutable in G, and also, in the case that $p = 2$, $k = 1$ and P is non-abelian, every cyclic subgroup of P of order 4 is S-propermutable in G. Then G is p-nilpotent.*

Proof. In fact, by Lemma 2, this theorem follows directly from the results by using the concept of Π-property, for example, the Main Theorem in [4]. $\square$

Clearly, propermutable subgroups are S-propermutable in G, and thus Problem 1 has a positive answer due to Theorem 3.

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